6. Gyakorlat

Feladat: megj.: az elemi fv-ek der.

$\boxed{1F.}$ (a) $(f(x))' = (e^{x^4})' = e^{x^4} \cdot 4x^3$

(b) $(f(x))' = (ch(e^{2x+3}))' = sh(e^{2x+3}) \cdot e^{2x+3} \cdot 2$

(c) $(f(x))' = \left(ln \frac{e^x}{1+e^x}\right)' = \frac{1}{\frac{e^x}{1+e^x}} \cdot \frac{e^x(1+e^x) - e^x \cdot e^x}{(1+e^x)^2}$

$= \dots$ HF.

(d) $f(x) = ln\left(\frac{1}{x} + \sqrt{1+\frac{1}{x^2}}\right)$

$D_f = ?$: $x \neq 0$; $\frac{1}{x} + \sqrt{1+\frac{1}{x^2}} > 0$

$\forall x \in \mathbb{R} \setminus \{0\}$ ✓

$\quad \| $

$\quad D_f$

$\forall x \in D_f$; $f \in D\{x\}$ ✓

$f'(x) = \frac{1}{\frac{1}{x} + \sqrt{1+\frac{1}{x^2}}} \cdot \left(\frac{1}{x} + \sqrt{1+\frac{1}{x^2}}\right)' =$ HF

$= \frac{1}{\frac{1}{x} + \sqrt{1+\frac{1}{x^2}}} \cdot \left(-\frac{1}{x^2} + \frac{1}{2\sqrt{1+\frac{1}{x^2}}} \cdot \left(\frac{1}{x^2}\right)'\right.$

$$\left(\frac{1}{x^2}\right)' = \left(x^{-2}\right)' = -2\cdot x^{-3} = -\frac{2}{x^3}$$

ÚTM.: $x>0$;
v. $x<0$;

$$\sqrt{1+\frac{1}{x^2}} = \sqrt{\frac{1+x^2}{x^2}} \overset{!}{=} \frac{\sqrt{1+x^2}}{|x|}$$

$\boxed{2.F.}$ $g(x)^{h(x)}$ típusú fv.

(a) $f(x) := x^x \quad (x>0)$.

ÖTLET: az alapot e-hatványként felírni.

$$\boxed{x = e^{\ln x} \quad x>0.}$$

$$g(x) = e^{\ln g(x)}$$

$f(x) = (e^{\ln x})^x = e^{x\cdot \ln x}$: összetett fv.

$\forall x>0$: $f\in D\{x\}$

$$f'(x) = e^{x\cdot\ln x}\cdot\left(1\cdot\ln x + x\cdot\frac{1}{x}\right) = x^x\cdot(\ln x+1)$$

(b) $f(x) = (\operatorname{tg} x)^x \quad \left(0<x<\frac{\pi}{2}\right)$

$/\operatorname{tg} x = \frac{\sin x}{\cos x} > 0/$

$$f(x) = (\operatorname{tg} x)^x = \left(e^{\ln \operatorname{tg} x}\right)^x = e^{x\ln\operatorname{tg} x}$$

$$f'(x) = e^{x\cdot\ln\operatorname{tg} x}\cdot\left(1\cdot\ln\operatorname{tg} x + x\cdot\frac{1}{\operatorname{tg} x}\cdot\frac{1}{\cos^2 x}\right) =$$

$$= (\operatorname{tg} x)^x \left(\ln \operatorname{tg} x + \frac{x}{\sin x \cdot \cos x}\right)$$

$$\frac{\sin 2x}{2}$$

$\boxed{3.F.}$ Egyoldali deriváltak

(a) $f(x) = x\cdot|x| \quad (x\in\mathbb{R})$.

abs $\notin D\{0\}$! De abs $\in D\{x\}$, ha $x\in\mathbb{R}\setminus\{0\}$.

Mj.: További fel. $\boxed{3.}$

$0 \qquad \infty$

$\forall x\in\mathbb{R}\setminus\{0\}$, $f\in D\{x\}$

$$f'(x) = \begin{cases} 2x, & \text{ha } x>0 \\ -2x, & \text{ha } x<0. \end{cases}$$

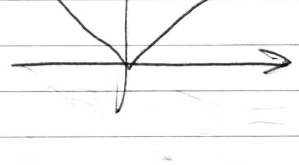
$a=0$:

1. lehetőség: a def.- $\displaystyle\lim_{x\to0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to0}|x|$

$\Rightarrow f\in D\{0\}$ és $f'(0)=0$.

2. lehetőség: egyoldali deriváltak:

abs.



$$\text{abs}'_+(0) = \lim_{x\to0+0}\frac{\overbrace{|x|}^{=x,\ \text{ha } x>0}-0}{x} = 1$$

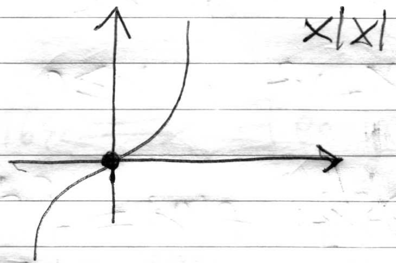
$$\text{abs}'_-(0) = \lim_{x\to0-0}\frac{|x|}{x} = \lim_{x\to0-0}(-1) = -1$$

$$\Rightarrow f'_+(0) = \lim_{x \to 0+0} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0+0} \frac{x^2-0}{x} = \underline{\underline{0}}.$$

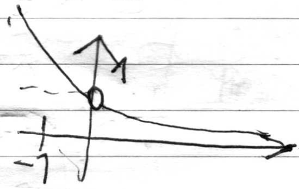
$$f'_-(0) = \lim_{x \to 0-0} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0-0} \frac{-x^2}{x} = \underline{\underline{0}}.$$

$$\Rightarrow f'_+(0) = f'_-(0) \Rightarrow f \in D\{0\}, \quad f'(0) = 0.$$

Mj.:



36. $$f(x) := \begin{cases} 1-x, & x<0 \\ e^{-x}, & x \geq 0 \end{cases}$$



Ha $x \in \mathbb{R}\setminus\{0\} \Rightarrow f \in D\{x\}$ és

$$f'(x) = \begin{cases} -1, & x<0 \\ -e^{-x}, & x>0. \end{cases}$$

a=0-ban:

Folytonos-e?

$$\lim_{x \to 0+0} f(x) = \lim_{x \to 0+0} e^{-x} = 1; \quad \lim_{x \to 0-0} f(x) = \lim_{x \to 0-0} 1-x = 1.$$

$$\Rightarrow f \in C\{0\}.$$

Deriválhatóság: e^{-x} deriválható α 0-ban $\Rightarrow \exists f'_+(0)$.

$$f'_+(0) = (e^{-x})'\Big|_{x=0} = -1$$

$$f'_-(0) = (1-x)'\Big|_{x=0} = -1$$

$\Rightarrow f \in D\{0\}$ és $f'(0) = -1$

$\boxed{3.c.}$ $f(x) := \begin{cases} \alpha x + x^2, & x < 0 \\ x - x^2, & x \geq 0 \end{cases}$

$\forall x \in \mathbb{R} \setminus \{0\}$: $f \in D\{x\}$; $f'(x) = \begin{cases} \alpha + 2x, & x < 0 \\ 1 - 2x, & x > 0 \end{cases}$

$\forall \alpha \in \mathbb{R}$ esetén!

$a = 0$: Folytonosság:

$\lim\limits_{x \to 0+0} f(x) = 0$, $\lim\limits_{x \to 0-0} f(x) = 0 \Rightarrow f \in C\{0\}$ $(\forall \alpha \in \mathbb{R})$.

Deriválhatóság: $f'_+(0) = 1$, $f'_-(0) = \alpha$

$f \in D\{0\} \Leftrightarrow 1 = \alpha$, és akkor $\underline{f'(0) = \alpha = 1}$

$\boxed{4.F.}$ $\lim\limits_{h \to 0} \dfrac{\sqrt[4]{16+h} - 2}{h} = ?$

1. lehetőség: Gyöktelenítés: $a^4 - b^4 = (a-b)(\ldots)$

2. új lehetőség: $f(x) := \sqrt[4]{x}$, $x > 0$.

$a = 16$

$f \in D\{16\}$, $f'(x) = (x^{1/4})' = \frac{1}{4} x^{-3/4} = \frac{1}{4} \cdot \frac{1}{(\sqrt[4]{x})^3}$

$f'(16) = \frac{1}{4} \cdot \frac{1}{2^3}$

$f \in D\{16\} \Longleftrightarrow \exists$ és véges: $\lim\limits_{h \to 0} \frac{f(16+h) - f(16)}{h} =$

$= \boxed{\lim\limits_{h \to 0} \frac{\sqrt{16-h} - \sqrt{16}}{h}}$ különbségi hányados határértéke 0-ban.

$\nexists \lim\limits_{h}$

$\boxed{6.F.}$ (b) $f(x) = \ln(1+x)$ $(x > -1)$

$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$

$f''(x) = (-1) \cdot (1+x)^{-2}$

$f'''(x) = 1 \cdot 2 \cdot (1+x)^{-3}$

$f^{(4)}(x) = -3!\,(1+x)^{-4}$

$\vdots$

$$\boxed{f^{(n)}(x) = (-1)^{n+1} \cdot (n-1)! \cdot \frac{1}{(1+x)^4} \quad (x > -1)}$$

Telj. ind.-val igazolás!

[illegible]: (a) $\sin x \quad (x \in \mathbb{R})$.

$\sin' x = \cos x$

$\sin'' x = -\sin x$

$\sin''' x = -\cos x$

$\sin^{(n)}(x) =$

Periodikusan!

$\sin^{IV} x = \sin x$

$\sin^{(5)}(x) \ldots$

$= \sin\left(x + \frac{n\pi}{2}\right).$