8. Gyakorlat

$\boxed{F}$ Igazoljuk: $\lim\limits_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e$.

Meo.: Em.l. $\lim\limits_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = e$

$$f(x) := \left(1+\frac{1}{x}\right)^x \quad (x>1)$$

$$f(x) = e^{x\ln\left(1+\frac{1}{x}\right)} \Longrightarrow f\in D.$$

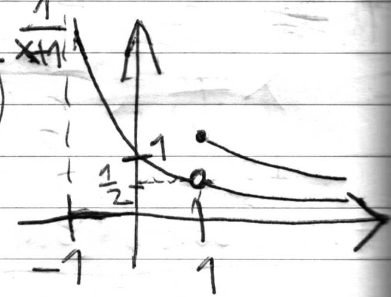
$$f'(x) = e^{x\ln\left(1+\frac{1}{x}\right)}\cdot\left(1\cdot\ln\left(1+\frac{1}{x}\right) + x\cdot\frac{1}{1+\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)\right) = \left(1+\frac{1}{x}\right)^x\cdot$$

$$\left(\ln\left(1+\frac{1}{x}\right)-\frac{1}{x+1}\right) \geq 0.$$

$$\Longleftrightarrow \ln\left(1+\frac{1}{x}\right)-\frac{1}{x+1}\geq 0, \quad x>1$$

$$\ln\left(1+\frac{1}{x}\right) \geq \frac{1}{x+1}$$

$$\ln 2 \geq \frac{1}{2}$$



$$g(x) := \ln\left(1+\frac{1}{x}\right)-\frac{1}{x+1} \quad (x\geq 1)$$

$$g(1)>0.$$

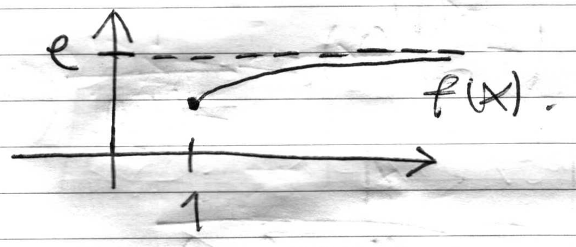
$$g'(x)>0 \Longrightarrow \ln\left(1+\frac{1}{x}\right)-\frac{1}{x+1}>0.$$

$$\forall x>1 \Longrightarrow f'(x)\geq 0 \quad \forall x>1 \Longleftarrow$$

$$\Rightarrow \boxed{f \uparrow (1,+\infty)} \Rightarrow \exists \lim_{x\to+\infty} f(x) \Rightarrow$$

átv. elv.

$$\underset{n\text{-é.-re}}{\Rightarrow} \lim_{n\to+\infty} f(n) = \lim_{n\to+\infty}\left(1+\frac{1}{n}\right) = e$$



Mj.: Másként:

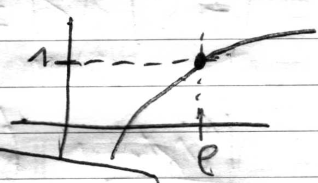
Ötlet: $\ln$, $\ln \in D$, $\ln' x = \frac{1}{x}$ $(x>0)$

$\Rightarrow$ $\ln' 1 = 1$

$$1 = \ln' 1 = \lim_{h\to 0}\frac{\ln(1+h) - \overbrace{\ln 1}^{=0}}{h} =$$

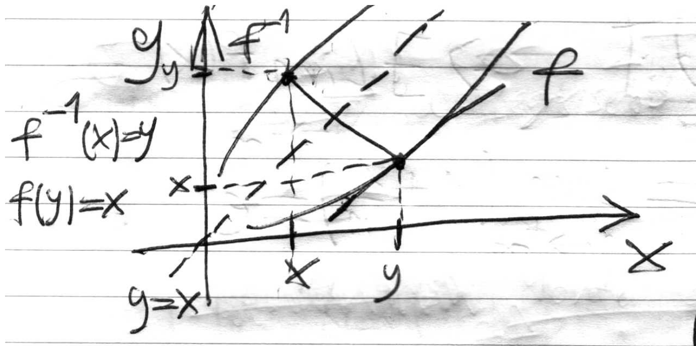
$$= \lim_{h\to 0} \ln(1+h)^{\frac{1}{h}} \underset{h=\frac{1}{x}}{=} \lim_{x\to+\infty} \ln\left(1+\frac{1}{x}\right)^x = 1$$

$\ln \in C\{e\}$



$$\Rightarrow \boxed{\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e}.$$

Eml.: Az inverz fv. deriválása.



Ha $f \in D\{y\}$
$f'(y) \neq 0$
$\Longrightarrow f^{-1} \in D\{x\}$

$$(f^{-1})'(x) = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}.$$

$\boxed{3.F.}$ $f(x) := x^3 + x$ $(x \in \mathbb{R})$

Ig. $\exists f^{-1}$; $f^{-1} \in D$
és szám. ki. $(f^{-1})'(2) = ?$

Meo. $f'(x) = 3x^2 + 1 > 0 \quad (\forall x \in \mathbb{R}) \Rightarrow$
$f \uparrow$ $\mathbb{R}$-en; $\Longrightarrow \boxed{\exists f^{-1}}$

Deriválhatóság: $f \in D(\mathbb{R})$, $f'(y) \neq 0$
$(\forall y \in \mathbb{R}) \Longrightarrow \forall x \in D_{f^{-1}}$, $f^{-1} \in D\{x\}$

⊛ $$(f^{-1})'(x) = \frac{1}{f'(y)}; \quad f(y) = x$$

Mj.: $(f^{-1})'(x)$ meghatározható $f^{-1}(x)$ explicit alakjának ismerete nélkül.

$\boxed{(f^{-1})'(2) = ?}$

$2 \overset{?}{\in} D_{f^{-1}} \Longleftrightarrow$

$\exists ?\ x \in \mathbb{R}:\ f(x) = x^3 + x = 2$

$x=1$; $\boxed{f(1)=2} \iff f^{-1}(2)=1$

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{4}.$$

■

HF. $\boxed{\boxed{GY.F.2.}}$

$\boxed{\text{TRIG. FV.-EK.}}$ (Részletek: L-Sch).

sin, cos ; tg, ctg.
hatványsorral.

T. A cos fv.-nek a $(0,2)$ int.-on pontosan egy gyöke van, azaz:

$$\exists!\ \alpha \in (0,2): \quad \cos\alpha = 0.$$

Legyen: $\boxed{\pi := 2\alpha}$

$\boxed{\text{Köv.}}$ $\cos\frac{\pi}{2}=0$; $\sin\frac{\pi}{2} = +1$;

(a) $\sin\pi = 0$; $\cos\pi = \cos^2\frac{\pi}{2} - \sin^2\frac{\pi}{2} = -1$

$\sin 2\pi = 0$; $\cos 2\pi = \cos^2\pi - \sin^2\pi = 1$

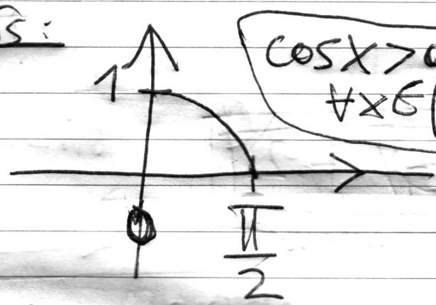
(b) $\sin(x+2\pi) = \sin x \cdot \cos 2\pi + \cos x \cdot \sin 2\pi$;
$(\forall x \in \mathbb{R})$ $= \sin x$

$$\cos(x+2\pi) = \cos x.$$

T A sin, cos 2π-szerint periodikus, és 2π a <u>legkisebb pozitív</u> periódus.

— o —

<u>Monotonitás:</u>



$\boxed{\cos x > 0 \quad \forall x \in \left(0, \frac{\pi}{2}\right)}$

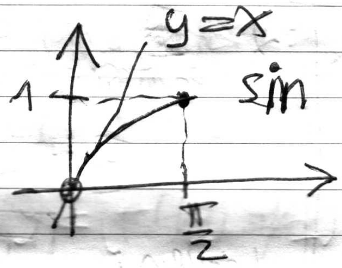
$$\sin' x = \cos x > 0 \quad \forall x \in \left(0, \frac{\pi}{2}\right)$$

$$\Rightarrow \sin \uparrow \quad \text{a } \left(0, \frac{\pi}{2}\right)\text{-en.}$$

$$\sin 0 = 0, \quad \sin \frac{\pi}{2} = 1$$

$$\sin' 0 = \cos 0 = 1$$

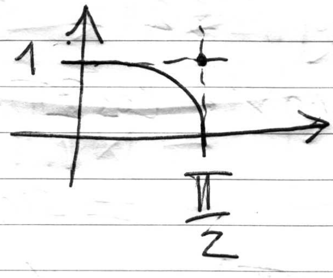
$$\sin' \frac{\pi}{2} = \cos \frac{\pi}{2} = 0.$$



— o —

$$\cos' x = -\sin x < 0 \quad \left(x \in \left(0, \frac{\pi}{2}\right)\right) \Rightarrow$$

$$\Rightarrow \cos \downarrow \quad (0, \pi/2)\text{-en.}$$

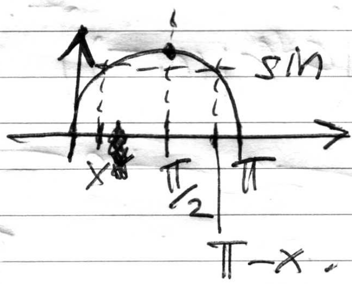


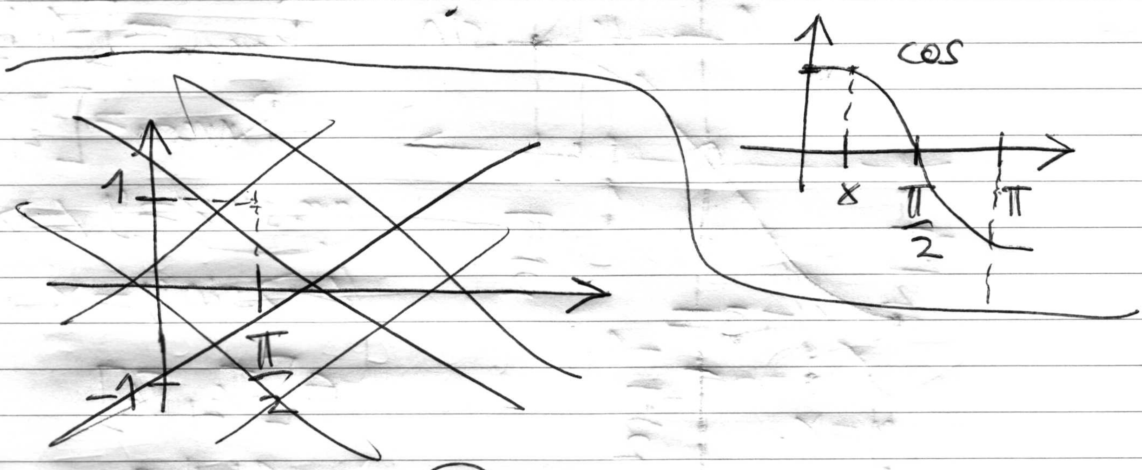
A többi intervallumon:

$$\sin(\pi - x) = \sin x$$

$$\cos(\pi - x) = -\cos x$$

⋮





tg, ctg (HF.) / ism.

$\boxed{\text{A trig. fv.-ek inverzeik.}}$

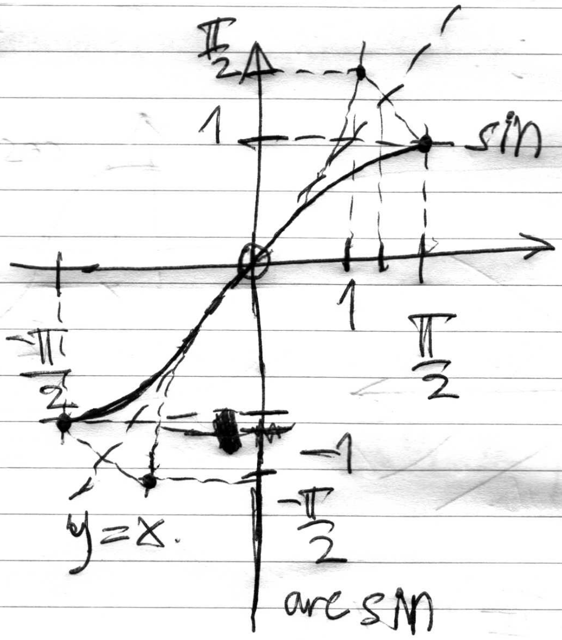
$$\left(\sin|_{[-\frac{\pi}{2}, \frac{\pi}{2}]}\right)^{-1} =: \arcsin$$

arkusszinusz-fv.

$$\left(\cos|_{[0,\pi]}\right)^{-1} =: \arccos.$$

$$\left(\mathrm{tg}|_{(-\frac{\pi}{2}, \frac{\pi}{2})}\right)^{-1} =: \mathrm{arc\,tg}.$$

$$\left(\mathrm{ctg}\,|\,(0,\pi)\right)^{-1} =: \mathrm{arc\,ctg}.$$



$D_{\arcsin} = [-1,1]$,

$R_{\arcsin} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\arcsin x = y \iff$

$\iff \sin y = x$

$\in \left[-\pi/2, \pi/2\right]$

Deriválhatóság:

$\sin' y \neq 0$

$y \in (-\pi/2, \pi/2)$

$\arcsin \in D\{x\} : |x| < 1$

$$\arcsin' x = \frac{1}{\sin' y} = \frac{1}{\cos y} = \pm\frac{1}{\sqrt{1-\sin^2 y}}$$

$\Rightarrow$ $x = \sin y$

$$\boxed{\arcsin' x = \frac{1}{\sqrt{1-x^2}} \quad (|x|<1).}$$

Hasonlóan: a többi trg. inverz.

— fv. képe

— tul.-i.

(pl. L-sch.) — deriváltt.